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Journal of Sound and Vibration 282 (2005) 1221–1230
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

On the eigenfrequencies of a cantilever beam carrying a tip spring–mass system with mass of the helical spring considered

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Received 27 January 2004; received in revised form 19 April 2004; accepted 29 April 2004

1. Introduction

In the investigation of the vibrations of elastic arms and their suppression in the field of mechanical engineering and robotics, various systems may be modeled as a clamped–free Bernoulli–Euler beam to which one or several helical spring–mass systems are attached. Some of the numerous publications on this subject are given in Refs. [1–4]. The common aspect of all these works is that the mass of the helical springs is not taken into account. It was observed that the effects of this assumption on the numerical values of the eigenfrequencies of these combined systems had not been investigated in the literature. As a first step to cover this gap, the frequency equation of the combined system described in Ref. [1] is derived, but with the helical spring having mass and without the added tip mass. To this end, the helical spring, as in literature [5], is modeled by an appropriate elastic rod that vibrates longitudinally.

The frequency equation obtained is solved numerically for various non-dimensional mass and spring parameters. Comparison with the massless spring case reveals that neglecting the mass can lead to serious errors for some parameter combinations.

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doi:10.1016/j.jsv.2004.04.020

2. Theory

The problem to be investigated in the present note is the natural vibration problem of the mechanical system shown in Fig. 1. It consists of a cantilevered Bernoulli–Euler beam to which an axially vibrating rod with a tip mass M is attached at the free end. It is assumed that this combined system vibrates only in the plane of the paper. The physical properties of the system are as follows. The length, mass per unit length and bending rigidity of the beam are L_1, m_1, E_1I_1 whereas the corresponding quantities and the axial rigidity of the rod are L_2, m_2, E_2A_2, respectively. It is to be noted that E_2A_2/L_2 corresponds to the spring constant of the helical spring.

The planar bending displacements of the beam in the coordinate system x_1, w_1 are denoted as $w_1(x_1,t)$, whereas the axial displacements of the rod are denoted as $w_2(x_2,t)$, where $x_2 = 0$ corresponds to the free end of the beam. Both $w_1(x_1,t)$ and $w_2(x_2,t)$ are assumed to be small.

In order to obtain the equations of motion of the system, Hamilton's principle

$$\int_{t_0}^{t_1} \delta(T - V)\,\mathrm{d}t = 0 \tag{1}$$

will be applied, where T and V denote the kinetic and potential energies, respectively. The total kinetic energy is

$$T = \tfrac{1}{2} m_1 \int_0^{L_1} \dot{w}_1^2(x_1, t)\,\mathrm{d}x_1 + \tfrac{1}{2} m_2 \int_0^{L_2} [\dot{w}_2(x_2, t) + \dot{w}_1(L_1, t)]^2\,\mathrm{d}x_2 + \tfrac{1}{2} M[\dot{w}_1(L_1, t) + \dot{w}_2(L_2, t)]^2. \tag{2}$$

The potential energy consists of two parts, one due to bending and the other due to axial displacements

$$V = \tfrac{1}{2} E_1 I_1 \int_0^{L_1} w''^2_1(x_1, t)\,\mathrm{d}x_1 + \tfrac{1}{2} E_2 A_2 \int_0^{L_2} w'^2_2(x_2, t)\,\mathrm{d}x_2. \tag{3}$$

In the above formulations, dots and primes denote partial derivatives with respect to time t and the position coordinate x_1 or x_2, respectively. After putting expressions (2) and (3) into expression

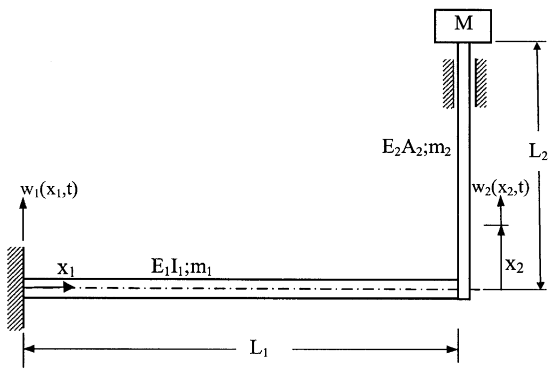

Fig. 1. Vibrational system under study: cantilevered beam carrying an axially vibrating rod with a tip mass.

(1) and carrying out the necessary variations, the following equations of motion of the beam and rod are obtained:

$$E_1I_1w_1^{iv}(x_1,t)+m_1\ddot{w}_1(x_1,t)=0,\quad E_2A_2w_2''(x_2,t)-m_2\ddot{w}_2(x_2,t)=m_2\ddot{w}_1(L_1,t). \tag{4,5}$$

The corresponding boundary and matching conditions for the two elastic domains are as follows

$$w_1(0,t)=0,\quad w_1'(0,t)=0,\quad w_1''(L_1,t)=0,\quad w_2(0,t)=0, \tag{6-9}$$

$$\int_0^{L_2} m_2[\ddot{w}_2(x_2,t)+\ddot{w}_1(L_1,t)]\,\mathrm{d}x_2+M[\ddot{w}_1(L_1,t)+\ddot{w}_2(L_2,t)]-E_1I_1w'''_1(L_1,t)=0, \tag{10}$$

$$M[\ddot{w}_1(L_1,t)+\ddot{w}_2(L_2,t)]+E_2A_2w_2'(L_2,t)=0. \tag{11}$$

The last two equations express the force balances at $x_1=L_1$ and $x_2=L_2$, respectively, whereas the meanings of the boundary conditions (6–9) are evident.

Using the standard method of separation of variables, one assumes

$$w_1(x_1,t)=W_1(x_1)\cos\omega t,\quad w_2(x_2,t)=W_2(x_2)\cos\omega t, \tag{12,13}$$

where $W_1(x_1)$ and $W_2(x_2)$ are the corresponding amplitude functions of the beam and the rod, respectively, and ω is the unknown eigenfrequency of the combined system. Substituting these expressions into the partial differential equations (4) and (5) results in the following ordinary differential equations:

$$W_1^{iv}(x_1)-\beta^4W_1(x_1)=0,\quad W_2''(x_2)+\gamma^2W_2(x_2)=-\gamma^2W_1(L_1). \tag{14,15}$$

Here, the following abbreviations are introduced:

$$\beta^4=m_1\omega^2/E_1I_1,\quad \gamma=\mu^2\beta^2,\quad \mu^4=\frac{E_1I_1m_2}{E_2A_2m_1}. \tag{16}$$

Now, the corresponding boundary conditions are

$$W_1(0)=0,\quad W_1'(0)=0,\quad W_1''(L_1)=0,\quad W_2(0)=0 \tag{17-20}$$

whereas the matching conditions (10) and (11) give

$$m_2\omega^2\int_0^{L_2}[W_2(x_2)+W_1(L_1)]\,\mathrm{d}x_2+M\omega^2[W_1(L_1)+W_2(L_2)]+E_1I_1W'''_1(L_1)=0, \tag{21}$$

$$-M\omega^2[W_1(L_1)+W_2(L_2)]+E_2A_2W_2'(L_2)=0. \tag{22}$$

Here, primes on $W_1(x_1)$ and $W_2(x_2)$ denote derivatives with respect to position coordinates x_1 and x_2, respectively.

The general solutions of the ordinary differential equations (14) and (15) are simply

$$W_1(x_1)=C_1\sin\beta x_1+C_2\cos\beta x_1+C_3\sinh\beta x_1+C_4\cosh\beta x_1, \tag{23}$$

$$W_2(x_2)=D_1\sin\gamma x_2+D_2\cos\gamma x_2-W_1(L_1), \tag{24}$$

where C_1–C_4 and D_1, D_2 are six integration constants to be evaluated via conditions (17–22).

The application of conditions (17–22) to solutions (23) and (24) yields a set of six homogeneous equations for the unknown coefficients C_1–C_4 and D_1, D_2. For a non-trivial solution to exist, the determinant of the coefficient matrix of this set of equations should be equal to zero which leads after some calculations to the following frequency equation of the vibrational system in Fig. 1:

$$[(\sin\bar{\beta}\cosh\bar{\beta}-\cos\bar{\beta}\sinh\bar{\beta})(\alpha_{11}\bar{\beta}^2+\sin\gamma L_2+\cos\gamma L_2-1)-\alpha_{22}\bar{\beta}(1+\cos\bar{\beta}\cosh\bar{\beta})] \cdot\left(\sin\gamma L_2-\frac{\alpha_{33}}{\bar{\beta}^2}\cos\gamma L_2\right)-(1-\cos\gamma L_2+\alpha_{11}\bar{\beta}^2\sin\gamma L_2)(\sin\bar{\beta}\cosh\bar{\beta}-\cos\bar{\beta}\sinh\bar{\beta}) \cdot\left(\cos\gamma L_2+\frac{\alpha_{33}}{\bar{\beta}^2}\sin\gamma L_2\right)=0. \tag{25}$$

Here, in addition to those given in Eq. (16), the following abbreviations are introduced:

$$\bar{\beta}=\beta L_1,\quad \bar{m}_{21}=m_2L_2/m_1L_1,\quad \alpha_k=(E_2A_2/L_2)/(E_1I_1/L_1^3),\quad \alpha_M=M/m_1L_1,$$
$$\gamma L_2=\bar{\beta}^2\sqrt{\frac{\bar{m}_{21}}{\alpha_k}},\quad \alpha_{11}=\frac{\alpha_M}{\sqrt{\bar{m}_{21}\alpha_k}},\quad \alpha_{22}=\frac{1}{\sqrt{\bar{m}_{21}\alpha_k}},\quad \alpha_{33}=\frac{\sqrt{\bar{m}_{21}\alpha_k}}{\alpha_M}. \tag{26}$$

Whence the roots $\bar{\beta}$ of Eq. (25) are calculated numerically, the eigenfrequencies ω of the vibrational system in Fig. 1 are obtained via Eq. (16) as

$$\omega=\bar{\beta}^2\sqrt{\frac{E_1I_1}{m_1L_1^4}}. \tag{27}$$

Recognizing that $\bar{m}_{21}$ denotes the ratio of the mass of the rod to that of the beam, it is reasonable to investigate as to which expression the frequency equation in Eq. (25) reduces for the limit $\bar{m}_{21}\to 0$. After some calculations and by using the well-known rule of L'Hospital, it can be shown that Eq. (25) reduces to

$$(\alpha_k-\alpha_M\bar{\beta}^4)(1+\cos\bar{\beta}\cosh\bar{\beta})+\alpha_k\alpha_M\bar{\beta}(\cos\bar{\beta}\sinh\bar{\beta}-\sin\bar{\beta}\cosh\bar{\beta})=0. \tag{28}$$

But this equation is just the frequency equation of a cantilevered Bernoulli–Euler beam carrying a massless spring–mass at the free end [6].

For getting trial values for the numerical solution of the frequency equation (25) on the one hand and for comparison purposes on the other, an approximate formula for the fundamental frequency of the system in Fig. 1 will be derived in the following.

According to Dunkerley's procedure, the mechanical system in Fig. 1 can be thought of as the "sum" of three subsystems shown in Fig. 2.

The fundamental frequency ω_{11} of the bare cantilevered Bernoulli–Euler beam is

$$\omega_{11}=\bar{\beta}_{11}^2\sqrt{\frac{E_1I_1}{m_1L_4^1}},\quad \bar{\beta}_{11}=1.87510407. \tag{29}$$

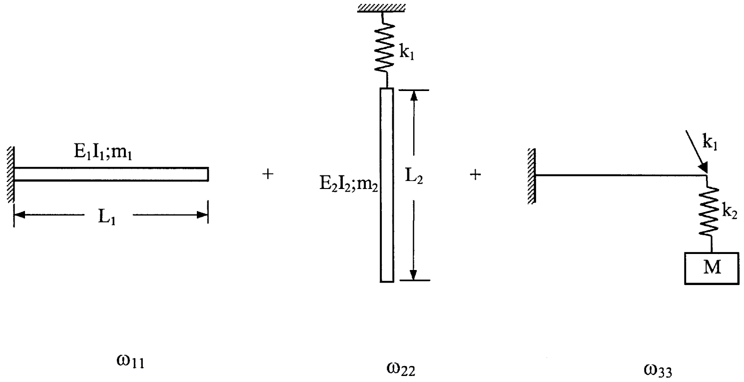

Fig. 2. Partial systems for the application of Dunkerley's procedure.

Making use of the more general expression in Ref. [7], the frequency equation of the second subsystem in Fig. 2 can be shown to be

$$\tan \bar{\beta}_{22} - \frac{3}{\alpha_k \bar{\beta}_{22}} = 0. \tag{30}$$

Once this transcendental equation is solved with respect to $\bar{\beta}_{22}$, the fundamental frequency of the second subsystem ω_{22} is simply

$$\omega_{22} = \bar{\beta}_{22} \sqrt{\frac{E_2 A_2}{m_2 L_2^2}}. \tag{31}$$

Finally, the eigenfrequency of the third subsystem in Fig. 2 can be obtained as

$$\omega_{33} = \sqrt{\frac{k_1 k_2}{M(k_1 + k_2)}}, \tag{32}$$

where the following spring constants

$$k_1 = \frac{3E_1 I_1}{L_1^3}, \quad k_2 = \frac{E_2 A_2}{L_2} \tag{33}$$

are introduced. As is known, k_1 represents the tip stiffness of the cantilevered beam.

According to Dunkerley's method, the approximate value of the fundamental frequency of the vibrational system in Fig. 1, ω_1 is obtained via

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} + \frac{1}{\omega_{33}^2}. \tag{34}$$

Substitution of the eigenfrequencies given in Eqs. (29),(31) and (32) into (34) yields

$$\omega_1 = \sqrt{\frac{\bar{\beta}_{11}^4}{1 + \frac{\bar{m}_{21}}{\alpha_k} \frac{\bar{\beta}_{11}^4}{\bar{\beta}_{22}^2} + \bar{\beta}_{11}^4 \frac{\alpha_M (3 + \alpha_k)}{3\alpha_k}}}\, \omega_0, \tag{35}$$

where $\omega_0^2 = E_1 I_1 / m_1 L_1^4$ is introduced. It is worth noting that $\bar{\beta}_{22}$ represents the first root of Eq. (30) for the corresponding α_k value.

In case of larger values of the non-dimensional stiffness parameter α_k, the first root of Eq. (30) can be given approximately but in a very accurate form. To this end, similar to the approach in Ref. [8], the approximation

$$\tan\,\bar{\beta}_{22} \approx \bar{\beta}_{22} + \frac{\bar{\beta}_{22}^3}{3} + \frac{2}{15}\,\bar{\beta}_{22}^5 \tag{36}$$

is made. After substitution of this into Eq. (30), one obtains

$$\bar{\beta}_{22} = \sqrt{\frac{\eta}{1 + \frac{\bar{\beta}_{22}^2}{3} + \frac{2}{15}\,\bar{\beta}_{22}^4}}, \tag{37}$$

where $\eta = 3/\alpha_k$ is introduced. The first approximation $\bar{\beta}_{22} \approx \sqrt{\eta}$ obtained from Eq. (30) yields after substitution into the right-hand side of Eq. (37)

$$\bar{\beta}_{22} = \sqrt{\frac{\eta}{1 + \frac{\eta}{3} + \frac{2}{15}\,\eta^2}}. \tag{38}$$

Hence, for large values of α_k, formula (35) based on Dunkerley's procedure can be reformulated as

$$\omega_1 = \sqrt{\frac{3\alpha_k \bar{\beta}_{11}^4}{3\alpha_k + \bar{m}_{21}\bar{\beta}_{11}^4\left(1 + \alpha_k + \frac{6}{5\alpha_k}\right) + \alpha_M \bar{\beta}_{11}^4 (3 + \alpha_k)}}\,\omega_0 \tag{39}$$

which can be evaluated directly for given values of the non-dimensional stiffness and mass parameters α_k and α_M, without having to solve an additional transcendental equation as before.

3. Numerical results

This section is devoted to the numerical evaluation of the formulas established in the preceding section. Recognizing that $\bar{m}_{21} = 0$ corresponds to the case of the mass of the axially vibrating rod, i.e., helical spring being zero, it is reasonable to make a comparison with the numerical results collected in Table 3 of Ref. [1] in which mass of the spring was neglected. The non-dimensional spring and mass parameters α_{ke} and α_{me} correspond to α_k and α_M here, respectively.

Table 1 reflects numerical results for the non-dimensional fundamental frequency parameter ω_1/ω_0 of the system in Fig. 1 taken from Ref. [1] and from the solution of Eq. (25) in connection with Eq. (27) and results obtained from Dunkerley-based formula (35). All numerical calculations were carried out with MATLAB.

The first figure in the column reading downwards is the fundamental frequency parameter ω_1/ω_0 taken from the second rows of Table 3 in Ref. [1], corresponding to $\bar{m}_{21} = 0$ here; the

Table 1
Non-dimensional fundamental frequency parameters ω_1/ω_0 of the system in Fig. 1 for various values of the non-dimensional spring and mass ratios

	α_k					
α_M	0.5	1	5	10	50	100
	0.921122	1.205409	1.764572	1.883690	1.988822	2.002508
	−0.0039	−0.0045	−0.0063	−0.0065	−0.0066	−0.0066
	0.917497	1.199944	1.753515	1.871398	1.975636	1.989208
	−0.0375	−0.0423	−0.0551	−0.0573	−0.0594	−0.0599
0.5	0.886569	1.154424	1.667356	1.775722	1.870729	1.882549
	−0.0323	−0.0450	−0.0442	−0.0327	−0.0157	−0.0129
	0.891388	1.151183	1.686505	1.822010	1.957560	1.976717
	−0.0682	−0.0827	−0.0900	−0.0818	−0.0688	−0.0666
	0.858323	1.105744	1.605839	1.729581	1.851979	1.869160
	0.653037	0.859420	1.307155	1.419384	1.527404	1.542201
	−0.0020	−0.0023	−0.0034	−0.0037	−0.0039	−0.0040
	0.651748	0.857457	1.302682	1.414102	1.521381	1.536077
	−0.0192	−0.0220	−0.0316	−0.0342	−0.0367	−0.0371
1	0.640468	0.840520	1.265828	1.370787	1.471349	1.484919
	−0.0167	−0.0240	−0.0270	−0.0210	−0.0103	−0.0084
	0.642134	0.838810	1.271800	1.389563	1.511643	1.529276
	−0.0361	−0.0450	−0.0543	−0.0508	−0.0432	−0.0418
	0.629437	0.820734	1.236118	1.347233	1.461410	1.477811
	0.292628	0.386724	0.606746	0.670001	0.736996	0.746678
	−0.00039	−0.00046	−0.00073	−0.00083	−0.00092	−0.00080
	0.292513	0.386546	0.606303	0.669445	0.736315	0.746079
	−0.0039	−0.0045	−0.0071	−0.0081	−0.0090	−0.0091
5	0.291477	0.384965	0.602423	0.664596	0.730327	0.739910
	−0.0034	−0.0050	−0.0064	−0.0052	−0.0027	−0.0020
	0.291624	0.384769	0.602849	0.666484	0.735024	0.745165
	−0.0076	−0.0097	−0.0129	−0.0125	−0.0108	−0.0103
	0.290406	0.382979	0.598922	0.661645	0.729023	0.738974
	0.206969	0.273659	0.431021	0.477054	0.526491	0.533782
	−0.00019	−0.00023	−0.00037	−0.00042	−0.00047	−0.00048
	0.206929	0.273596	0.430862	0.476853	0.526242	0.533526
	−0.0020	−0.0023	−0.0036	−0.0041	−0.0047	−0.0047
10	0.206561	0.273034	0.429458	0.475082	0.524034	0.531249
	−0.0017	−0.0025	−0.0033	−0.0027	−0.0014	−0.0011
	0.206613	0.272963	0.429606	0.475765	0.525761	0.533184
	−0.0038	−0.0049	−0.0066	−0.0064	−0.0056	−0.0054
	0.206179	0.272321	0.428178	0.473995	0.523551	0.530903
	0.092578	0.122457	0.193471	0.214535	0.237417	0.240821
	−0.000043	−0.000049	−0.000072	−0.000084	−0.000093	−0.000100
	0.092574	0.122451	0.193457	0.214517	0.237395	0.240797
	−0.00043	−0.00047	−0.00074	−0.00084	−0.00096	−0.00098

Table 1 (*continued*)

	α_k					
α_M	0.5	1	5	10	50	100
50	0.092541	0.122400	0.193328	0.214354	0.237190	0.240586
	−0.00036	−0.00051	−0.00067	−0.00055	−0.00028	−0.00023
	0.092545	0.122394	0.193342	0.214416	0.237350	0.240765
	−0.00078	−0.00099	−0.0013	−0.0013	−0.0011	−0.0011
	0.092506	0.122336	0.193211	0.214254	0.237146	0.240554
	0.065464	0.086596	0.136867	0.151805	0.168056	0.170475
	−0.000031	−0.000023	−0.000037	−0.000040	−0.000054	−0.000053
	0.065462	0.086594	0.136862	0.151799	0.168047	0.170466
	−0.00020	−0.00023	−0.00037	−0.00042	−0.00049	−0.00049
100	0.065451	0.086576	0.136817	0.151741	0.167974	0.170391
	−0.00018	−0.00025	−0.00033	−0.00021	−0.00015	−0.00012
	0.065452	0.086574	0.136822	0.151763	0.168031	0.170455
	−0.00038	−0.00050	−0.00067	−0.00066	−0.00058	−0.00056
	0.065439	0.086553	0.136775	0.151705	0.167959	0.170380

The first figure reading downwards is the fundamental frequency parameter taken from Ref. [1], the second and third are obtained from Eq. (25) for $\bar{m}_{21} = 0.01$, 0.1 respectively. The fourth and fifth are Dunkerley-based values obtained from formula (35) also for $\bar{m}_{21} = 0.01$ and 0.1, respectively. The right-shifted values indicate relative errors with respect to the first-row values.

Table 2
Numerical solutions of Eq. (30) especially for large α_k values and approximate solutions given by Eq. (38)

α_k	$\bar{\beta}_{22}$ (from Eq. (30))	$\bar{\beta}_{22}$ (from Eq. (38))
0.5	1.34955282	0.87705802
1	1.19245883	0.96824584
5	0.70506550	0.69337525
10	0.52179118	0.51940752
50	0.24252625	0.24247858
100	0.17234380	0.17233526
200	0.12216914	0.12216762
300	0.09983364	0.09983309
400	0.08649444	0.08649417
500	0.07738229	0.07738214
600	0.07065181	0.07065171
700	0.06541864	0.06541858
800	0.06119900	0.06119895
900	0.05770297	0.05770294
1000	0.05474488	0.05474486

second and third are frequency parameters obtained from Eq. (25) for $\bar{m}_{21} = 0.01$, 0.1, respectively, whereas the fourth and fifth are approximate values coming from the Dunkerley-based formula (35), for $\bar{m}_{21} = 0.01$ and 0.1, respectively.

It is worth noting that the first figures can be obtained from Eq. (28) as well, as this equation is equivalent to that given in Refs. [1,6] for the special case $\bar{m}_{21} = 0$. The right-shifted values in each cell indicate the relative errors with respect to the first-row values.

As expected, the values given in the fourth and fifth rows which are based on Dunkerley's formula, are smaller than those in the second and third rows; indeed, it is a known fact that the Dunkerley-based values are always smaller than the actual values. These values constitute very suitable initial values in the numerical solution of the transcendental Eq. (25) from which the second- and third-row values are obtained.

It is natural that the values in the second and third rows are smaller than the first-row values which correspond to the massless spring case, because these correspond to the case with the spring with mass. On the other hand, the third-row values are smaller than the second-row values, because the spring mass in the third row is 10 times the one in the second row. In harmony with this fact, the fifth-row values are smaller than the fourth-row values. It can be said that the relative errors corresponding to $\bar{m}_{21} = 0.1$ are approximately 10 times those for the case $\bar{m}_{21} = 0.01$.

Moving downward in a single column in the table, i.e., as α_M gets larger, the relative errors get smaller. On the contrary, for a fixed α_M value, the errors increase in general towards the larger α_k values. Thus, Table 1 leads to the conclusion that neglecting the spring mass will result in larger errors especially in the small α_M and large α_k regions.

In Table 2, the numerical solutions of Eq. (30), especially for large α_k values are combined with the approximate solutions given in Eq. (38). It is clearly seen that formula (38) gives accurate approximate values as α_k gets larger.

4. Conclusions

As the model of many actual systems in the literature, Bernoulli–Euler beams with various supporting conditions and spring–mass additions are used. However, in these applications the helical springs are frequently assumed to be massless. In the present study, a helical spring with mass is modeled as a longitudinally vibrating rod, attached to the tip of a cantilevered beam together with an additional mass, thus composing the system under study. The frequency equation of this combined system is derived. Comparison of the numerical results with the massless spring case reveals the fact that neglecting the mass can lead to serious errors for some parameter combinations.

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